Lecture 2 — Model Categories: The Homotopy Category

Proposition 1. *Let C be a model category. The inclusion functors between full subcategories*

$$C_{cf} \hookrightarrow C_c \xrightarrow{\;\;i\;\;} C$$
$$\searrow \qquad \nearrow$$
$$C_f$$

induce equivalences of categories

$$C_{cf}[W^{-1}] \xrightarrow{\;\sim\;} C_c[W^{-1}] \xrightarrow{\;\sim\;} C[W^{-1}],$$

and analogously $C_{cf}[W^{-1}] \xrightarrow{\sim} C_f[W^{-1}] \xrightarrow{\sim} C[W^{-1}]$.

Proof. We show that $C_c[W^{-1}] \to C[W^{-1}]$ is an equivalence; the remaining cases are dual or analogous.

The functor. Since the inclusion $i\colon C_c \hookrightarrow C$ preserves weak equivalences, it induces a functor $\widetilde{i}\colon C_c[W^{-1}] \to C[W^{-1}]$ on the localised categories.

The inverse. The cofibrant replacement functor $Q\colon C \to C_c$, $x \mapsto Qx$, preserves weak equivalences. The natural map $q_x\colon Qx \xrightarrow{\sim} x$ is a trivial fibration, hence a weak equivalence, so Q induces a functor $\widetilde{Q}\colon C[W^{-1}] \to C_c[W^{-1}]$.

Natural isomorphisms. The natural transformation $iQ \Rightarrow \mathrm{Id}_C$ given by q, and the natural transformation $Qi \Rightarrow \mathrm{Id}_{C_c}$ (using that $q_{ix}\colon Q(ix) \xrightarrow{\sim} ix$ is a weak equivalence), become natural isomorphisms in the localised categories. Hence $\widetilde{i}$ and $\widetilde{Q}$ are inverse equivalences. $\qquad\square$

Definition 2. Let C be a model category and let $f, g\colon B \to X$ be two maps.

1. A **cylinder object for** B is a factorisation of the fold map $\nabla\colon B \sqcup B \to B$ as

$$B \sqcup B \underset{\in \mathrm{Cof}}{\xrightarrow{(i_0, i_1)}} B' \xrightarrow[\sim]{\;s\;} B,$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{\nabla}$$

   where $(i_0, i_1) \in \mathrm{Cof}$ and $s\colon B' \to B$ lies in W.

2. A **path object (co-cylinder) for** X is a factorisation of the diagonal $\Delta\colon X \to X \times X$ as

$$X \xrightarrow[\sim]{\;r\;} X' \underset{\in \mathrm{Fib}}{\xrightarrow{(p_0, p_1)}} X \times X,$$
$$\underbrace{\qquad\qquad\qquad\qquad}_{\Delta}$$

   where $r\colon X \to X'$ lies in W and $(p_0, p_1) \in \mathrm{Fib}$.

3. A **left homotopy** $f \overset{l}{\sim} g$ is a map $H\colon B' \to X$ for some cylinder object B' of B, satisfying $Hi_0 = f$ and $Hi_1 = g$.

4. A **right homotopy** $f \overset{r}{\sim} g$ is a map $K\colon B \to X'$ for some path object X' of X, satisfying $p_0 K = f$ and $p_1 K = g$.

5. We say that f and g are **homotopic**, written $f \sim g$, if $f \overset{l}{\sim} g$ and $f \overset{r}{\sim} g$.

6. We say that f is a **homotopy equivalence** if there exists $h\colon X \to B$ such that $hf \sim \mathrm{Id}_B$ and $fh \sim \mathrm{Id}_X$.

Remark. *A path object for X in C is exactly a cylinder object for X in the opposite model category C^{op}. In particular,*

$$f \overset{l}{\sim} g \text{ in } C \iff f \overset{r}{\sim} g \text{ in } C^{\mathrm{op}}.$$

It is therefore sufficient to prove results about left homotopies; the dual statements about right homotopies follow automatically.

Functorial cylinder and path objects. By applying the functorial factorisations to ∇ and Δ respectively, one obtains a functorial cylinder $\mathrm{Cyl}(B)$ and a functorial path object $\mathrm{Path}(X)$:

Equivalence of cylinder and path objects. Every cylinder object B' for B is weakly equivalent to $\mathrm{Cyl}(B)$. More precisely, there exists a weak equivalence $h\colon B' \to \mathrm{Cyl}(B)$ compatible with all structure maps, as recorded by the commutative diagram

If $t\colon \mathrm{Cyl}(B) \to B$ denotes the functorial projection, then $s = t \circ h$. Since $s \in W$ and $t \in W$, the two-out-of-three axiom gives $h \in W$.

Dually, every path object X' for X is weakly equivalent to $\mathrm{Path}(X)$.

Proposition 3. *Let C be a model category and let $f, g\colon B \to X$.*

1. *If $f \overset{l}{\sim} g$ and $h\colon X \to Y$, then $hf \overset{l}{\sim} hg$.*

   Dual: if $f \overset{r}{\sim} g$ and $h\colon A \to B$, then $fh \overset{r}{\sim} gh$.

2. *If X is fibrant, $f \overset{l}{\sim} g$, and $h\colon A \to B$, then $fh \overset{l}{\sim} gh$.*

   Dual: if B is cofibrant, $f \overset{r}{\sim} g$, and $h\colon X \to Y$, then $hf \overset{r}{\sim} hg$.

3. *If B is cofibrant, then left homotopy is an equivalence relation on $C(B,X)$.*

   Dual: if X is fibrant, then right homotopy is an equivalence relation on $C(B,X)$.

4. *If B is cofibrant and $h\colon X \to Y$ is a weak equivalence satisfying either*

$$h \in W \cap \mathrm{Fib}, \qquad or \qquad h \in W \ \text{with}\ X, Y \in C_f,$$

   then h induces an isomorphism $C(B,X)/\overset{l}{\sim} \ \overset{\sim}{\longrightarrow}\ C(B,Y)/\overset{l}{\sim}$.

   Dual: if X is fibrant and $h \in W \cap \mathrm{Cof}$, or $h \in W$ with $A, B \in C_c$, then h induces an analogous isomorphism for $\overset{r}{\sim}$.

5. *If B is cofibrant, then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$. Furthermore, for any path object X' of X there exists a right homotopy $K\colon B \to X'$ from f to g.*

   Dual: if X is fibrant, then $f \overset{r}{\sim} g$ implies $f \overset{l}{\sim} g$, and the left homotopy can be realised using any cylinder object for B.

Proof of Proposition 3. By the duality remark it suffices to prove all claims about left homotopies.

Part (1). Given a left homotopy $H\colon B' \to X$ with $Hi_0 = f$ and $Hi_1 = g$, the composite $hH\colon B' \to Y$ satisfies $(hH)i_0 = hf$ and $(hH)i_1 = hg$, so it is a left homotopy from hf to hg.

Part (2). Let X be fibrant, let $H\colon B' \to X$ be a left homotopy from f to g via the cylinder (B', i_0, i_1, s), and let $h\colon A \to B$.

Step 1: improving the cylinder. Factor the projection $s\colon B' \to B$ as

$$
\begin{array}{ccc}
 & B'' & \\
{\scriptstyle s' \,\in W\cap\mathrm{Cof}}\nearrow & & \searrow{\scriptstyle s'' \,\in W\cap\mathrm{Fib}} \\
B' & \longrightarrow & B.
\end{array}
$$

Since X is fibrant, the trivial cofibration $s'\colon B' \to B''$ admits an extension of the homotopy:

$$
\begin{array}{ccc}
B' & \overset{H}{\longrightarrow} & X \\
{\scriptstyle s' \,\in W\cap\mathrm{Cof}}\downarrow & \overset{\widetilde{H}}{\nearrow} & \downarrow \\
B'' & \longrightarrow & *.
\end{array}
$$

The map $\widetilde{H}\colon B'' \to X$ is a left homotopy from f to g through the cylinder B'' with projection $s''\colon B'' \to B \in W \cap \mathrm{Fib}$.

Step 2: precomposing with h. Let $A \sqcup A \overset{j}{\to} A' \overset{t}{\to} A$ be a cylinder object for A with $j \in \mathrm{Cof}$. Consider the lifting problem

$$
\begin{array}{ccc}
A \sqcup A & \overset{i\circ(h\sqcup h)}{\longrightarrow} & B'' \\
{\scriptstyle j \,\in\mathrm{Cof}}\downarrow & \overset{K}{\nearrow} & \downarrow{\scriptstyle s'' \,\in W\cap\mathrm{Fib}} \\
A' & \underset{h\circ t}{\longrightarrow} & B.
\end{array}
$$

The square commutes: $s'' \circ i \circ (h \sqcup h) = h \sqcup h = (h \circ t) \circ j$. Since $j \in \mathrm{Cof}$ and $s'' \in W \cap \mathrm{Fib}$, a lift $K \colon A' \to B''$ exists. The composite $\widetilde{H} \circ K \colon A' \to X$ is a left homotopy from fh to gh:

$$(\widetilde{H}K)j_0 = \widetilde{H}(i_0 h) = fh, \qquad (\widetilde{H}K)j_1 = \widetilde{H}(i_1 h) = gh.$$

Part (3): left homotopy is an equivalence relation (assuming B cofibrant).

Reflexivity. For any cylinder B' with projection $s \colon B' \to B$, the composite $f \circ s \colon B' \to X$ satisfies $(fs)i_0 = f(si_0) = f$ and $(fs)i_1 = f(si_1) = f$, hence it is a left homotopy from f to itself. No cofibrancy assumption is needed.

Symmetry. Given a left homotopy $H \colon B' \to X$ from f to g, swap the roles of i_0 and i_1 in B' to get a new cylinder B^{op}; then H witnesses $g \overset{l}{\sim} f$. No cofibrancy assumption is needed.

Transitivity (requires B cofibrant). Let $H \colon B' \to X$ be a left homotopy from f to g, and let $H' \colon B'' \to X$ be a left homotopy from g to h. Form the pushout

$$
\begin{array}{ccc}
B & \xrightarrow{\;i_1^{B'}\;} & B' \\
{\scriptstyle i_0^{B''}}\big\downarrow & & \big\downarrow \\
B'' & \longrightarrow & C.
\end{array}
$$

Set $j_0 \colon B \xrightarrow{i_0^{B'}} B' \to C$ and $j_1 \colon B \xrightarrow{i_1^{B''}} B'' \to C$, and let $t \colon C \to B$ be induced by the cylinder projections $s \colon B' \to B$ and $s' \colon B'' \to B$. This gives a factorisation $B \sqcup B \xrightarrow{j_0 \sqcup j_1} C \xrightarrow{t} B$ of the fold map.

The map $j_0 \sqcup j_1$ need not be a cofibration, but t is a weak equivalence. Indeed, since B is cofibrant the map $i_1^{B'} \colon B \to B'$ lies in $W \cap \mathrm{Cof}$; by the pushout axiom the map $B'' \to C$ also lies in $W \cap \mathrm{Cof}$. Since the composite $B'' \to C \xrightarrow{t} B$ equals $s' \in W$, the two-out-of-three axiom gives $t \in W$.

We factor $j_0 \sqcup j_1$ into a cofibration followed by a trivial fibration to obtain a cylinder object C' for B:

$$
B \sqcup B \xrightarrow{j_0 \sqcup j_1} C \xrightarrow{\;t \sim\;} B.
$$

The map $K \colon C \to X$ induced by H and H' satisfies $Kj_0 = f$ and $Kj_1 = h$. The composite $C' \to C \xrightarrow{K} X$ is a left homotopy $f \overset{l}{\sim} h$ through the cylinder C'.

Part (4): h induces an isomorphism. We treat the two cases separately.

Case 1: $h \in W \cap \mathrm{Fib}$, with B cofibrant. The map $h_* \colon C(B,X)/\overset{l}{\sim} \to C(B,Y)/\overset{l}{\sim}$ is well defined by part (1) and the quotients exist by part (3).

Surjectivity. Since B is cofibrant and $h \in W \cap \mathrm{Fib}$, a lift exists in

$$
\begin{array}{ccc}
\varnothing & \longrightarrow & X \\
{\scriptstyle \in \mathrm{Cof}}\big\downarrow & {\scriptstyle f}\nearrow & \big\downarrow {\scriptstyle h\, \in W \cap \mathrm{Fib}} \\
B & \xrightarrow{\;f'\;} & Y,
\end{array}
$$

so every $f' \colon B \to Y$ lifts to an $f \colon B \to X$ with $hf = f'$, and h_* is surjective.

Injectivity. Suppose $hf \overset{l}{\sim} hg$ via a left homotopy $H \colon B' \to Y$. Since B is cofibrant and $h \in W \cap \mathrm{Fib}$, a lift K exists in

$$
\begin{array}{ccc}
B \sqcup B & \xrightarrow{(f \sqcup g)} & X \\
{\scriptstyle \in \mathrm{Cof}} \downarrow & \overset{K}{\nearrow} & \downarrow {\scriptstyle h \, \in W \cap \mathrm{Fib}} \\
B' & \xrightarrow[H]{} & Y.
\end{array}
$$

The lift satisfies $Ki_0 = f$ and $Ki_1 = g$, so $f \overset{l}{\sim} g$.

Case 2: $h \in W$ with $X, Y \in C_f$. Apply Ken Brown's Lemma in the following form: if a functor F from a model category sends every trivial fibration between fibrant objects to an isomorphism, then it sends every weak equivalence between fibrant objects to an isomorphism. Take $F = C(B, -)/\overset{l}{\sim} \colon C_f \to \mathrm{Set}$. Case 1 shows F sends trivial fibrations between fibrant objects to isomorphisms, so Ken Brown's Lemma gives that $F(h)$ is an isomorphism for every $h \in W$ with $X, Y \in C_f$.

Part (5): if B is cofibrant then $f \overset{l}{\sim} g$ implies $f \overset{r}{\sim} g$ through all the path objects for X

Let $H \colon B' \to X$ be a left homotopy from f to g via the cylinder (B', i_0, i_1, s). Since B is cofibrant, $s \in W$, and $s \circ i_0 = \mathrm{Id}_B \in W$, the two-out-of-three axiom gives $i_0 \in W$; combined with $i_0 \in \mathrm{Cof}$ this yields $i_0 \colon B \to B' \in W \cap \mathrm{Cof}$. Let $X \overset{r}{\to} X' \xrightarrow{(p_0, p_1)} X \times X$ be a path object for X. Solve the lifting problem

$$
\begin{array}{ccc}
B & \xrightarrow{rf} & X' \\
{\scriptstyle i_0 \, \in W \cap \mathrm{Cof}} \downarrow & \overset{J}{\nearrow} & \downarrow {\scriptstyle (p_0, p_1) \, \in \mathrm{Fib}} \\
B' & \xrightarrow[(f \circ s, \, H)]{} & X \times X.
\end{array}
$$

The square commutes: $(p_0, p_1) \circ rf = (p_0 r \circ f, \, p_1 r \circ f) = (f, f)$ (since $p_0 r = p_1 r = \mathrm{Id}_X$); and $(f \circ s, H) \circ i_0 = (f \circ s \circ i_0, \, H \circ i_0) = (f, f)$ (since $si_0 = \mathrm{Id}_B$ and $Hi_0 = f$). Setting $K := Ji_1 \colon B \to X'$ and using $p_0 J = f \circ s$ and $p_1 J = H$:

$$
p_0 K = p_0 J i_1 = (f \circ s)(i_1) = f(s \, i_1) = f, \qquad p_1 K = p_1 J i_1 = H \, i_1 = g.
$$

Hence K is a right homotopy from f to g. $\qquad\qquad\square$

Corollary 4. *If $B \in C_c$ and $X \in C_f$, then the left and right homotopy relations coincide and define a single equivalence relation $\sim$ on $C(B, X)$. Moreover, $f \sim g$ if and only if they are left homotopic (equivalently, right homotopic) through* any *cylinder object for B (resp. path object for X).*

Corollary 5. *The homotopy relation $\sim$ on the morphisms of C_{cf} is an equivalence relation compatible with composition. The quotient category $C_{cf}/\sim$ therefore exists, and the canonical functor $C_{cf} \to C_{cf}/\sim$ inverts all homotopy equivalences.*

Remark. *The canonical functor $C_{cf} \to C_{cf}/\sim$ inverts homotopy equivalences. The goal is to show it also inverts weak equivalences.*

Proposition 6. *Let C be a model category. A map in C_{cf} is a weak equivalence if and only if it is a homotopy equivalence.*

Proof. ($\Rightarrow$) Let $f\colon A \to B$ be a weak equivalence in C_{cf}.

By part (4) of Proposition 3 (applied with $X, Y \in C_f \supseteq C_{cf}$ and $h = f \in W$), for any $X \in C_{cf}$ the postcomposition map

$$f^*\colon (C_{cf}/\!\sim)(X, A) \longrightarrow (C_{cf}/\!\sim)(X, B)$$

is a bijection. Taking $X = B$ yields $g\colon B \to A$, unique up to homotopy, with $fg \sim \mathrm{Id}_B$. Since $fgf \sim f$, taking $X = A$ forces $gf \sim \mathrm{Id}_A$, so f is a homotopy equivalence.

($\Leftarrow$) Let $f\colon A \to B$ be a homotopy equivalence in C_{cf}.

Factor $f = p \circ g$ with $g \in W \cap \mathrm{Cof}$ and $p \in \mathrm{Fib}$:

$$
\begin{array}{ccc}
A & \xrightarrow{\;\;f\;\;} & B \\
 & \searrow_{g\,\in W \cap \mathrm{Cof}} \quad \nearrow_{p\,\in\mathrm{Fib}} & \\
 & C. &
\end{array}
$$

Since $g \in W$, the map g is a homotopy equivalence by the forward direction. Note that C is cofibrant (since A is cofibrant and $g \in W \cap \mathrm{Cof}$) and fibrant (since $p \in \mathrm{Fib}$ and B is fibrant, so $C \to *$ factors through p followed by $B \to * \in \mathrm{Fib}$, giving $C \in C_f$). Hence $C \in C_{cf}$.

It remains to show $p \in W$.

Let $f'\colon B \to A$ be a homotopy inverse of f, and let $H\colon B' \to B$ be a left homotopy from ff' to Id_B, with $i_0\colon B \to B' \in W \cap \mathrm{Cof}$ (which holds since $B \in C_{cf}$ is cofibrant and $si_0 = \mathrm{Id}_B \in W$). Solve the lifting problem

$$
\begin{array}{ccc}
B & \xrightarrow{\;gf'\;} & C \\
{\scriptstyle i_0\,\in W \cap \mathrm{Cof}}\big\downarrow & {}^{H'}\nearrow & \big\downarrow {\scriptstyle p\,\in\mathrm{Fib}} \\
B' & \xrightarrow{\;\;H\;\;} & B.
\end{array}
$$

Setting $q := H'i_1\colon B \to C$, we get $pq = H i_1 = \mathrm{Id}_B$, and H' is a left homotopy from gf' to q.

Let $g'\colon C \to A$ be a homotopy inverse of g. Then:

$$p \sim p \circ gg' \sim fg',$$
$$qp \sim (gf')(fg') = g(f'f)g' \sim g\,\mathrm{Id}_A\,g' = gg' \sim \mathrm{Id}_C.$$

Hence $qp \sim \mathrm{Id}_C$. If $K : C' \to C$ is a left homotopy $\mathrm{Id}_C \to qp$, $Ki_0 = \mathrm{Id}_C$ is a weak equivalence, as is i_0 and, so, $K \in W$. Thus $Ki_1 = qp$ is a weak equivalence.

To deduce $p \in W$, exhibit p as a retract of qp in the arrow category:

$$
\begin{array}{ccccc}
C & \xrightarrow{\;\mathrm{Id}_C\;} & C & \xrightarrow{\;\mathrm{Id}_C\;} & C \\
{\scriptstyle p}\big\downarrow & & {\scriptstyle qp}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
B & \xrightarrow{\;\;q\;\;} & C & \xrightarrow{\;\;p\;\;} & B.
\end{array}
$$

The horizontal composites are Id_C on top and $pq = \mathrm{Id}_B$ on the bottom, so p is a retract of qp in $\mathrm{Arr}(C)$. Since $qp \in W$ and W is closed under retracts, $p \in W$, hence $f = pg \in W$. $\qquad\square$

Corollary 7. *Let C be a model category, and let*

$$\gamma\colon C_{cf} \longrightarrow C_{cf}[W^{-1}], \qquad \delta\colon C_{cf} \longrightarrow C_{cf}/\sim$$

be the canonical functors. There exists a unique isomorphism of categories $j\colon C_{cf}/\sim \xrightarrow{\sim} C_{cf}[W^{-1}]$ such that $j \circ \delta = \gamma$; moreover, j is the identity on objects.

Proof. The functor δ sends homotopy equivalences to isomorphisms. By Proposition 6 it therefore sends weak equivalences to isomorphisms as well. The universal property of $\operatorname{Ho} C_{cf}$ then yields a unique functor $j\colon C_{cf}/\sim \to C_{cf}[W^{-1}]$ with $j\delta = \gamma$. The inverse is constructed by the same universal property applied to δ. $\square$

Definition. The **homotopy category** of C is defined as $\operatorname{Ho} C := C[W^{-1}]$.

Theorem 8. *Let C be a model category, let $\gamma\colon C \to \operatorname{Ho} C$ be the canonical functor, let Q denote the cofibrant replacement functor, and let R denote the fibrant replacement functor.*

1. *The inclusion $C_{cf} \hookrightarrow C$ induces equivalences of categories*

$$C_{cf}/\sim \xrightarrow{\ \sim\ } \operatorname{Ho} C_{cf} \xrightarrow{\ \sim\ } \operatorname{Ho} C.$$

2. *There are natural isomorphisms:*

   (a) *$C(QRX, QRY)/\sim\ \cong\ \operatorname{Ho} C(\gamma X, \gamma Y)\ \cong\ C(RQX, RQY)/\sim$.*
   (b) *$\operatorname{Ho} C(\gamma X, \gamma Y)\ \cong\ C(QX, RY)/\sim$.*
   (c) *If X is cofibrant and Y is fibrant, then $\operatorname{Ho} C(\gamma X, \gamma Y)\ \cong\ C(X, Y)/\sim$.*

3. *The functor $\gamma\colon C \to \operatorname{Ho} C$ identifies left and right homotopic maps: if $f \overset{l}{\sim} g$ or $f \overset{r}{\sim} g$, then $\gamma f = \gamma g$.*

4. *If $f\colon A \to B$ is a map in C such that γf is an isomorphism in $\operatorname{Ho} C$, then $f \in W$.*

Proof. **Part (1).** Combine Proposition 1 and Corollary 7.

Part (2). The inverse of the equivalence $\operatorname{Ho} C_{cf} \to \operatorname{Ho} C$ is given by $\operatorname{Ho} R \circ \operatorname{Ho} Q$ (equivalently, $\operatorname{Ho} Q \circ \operatorname{Ho} R$), yielding the isomorphisms in (a). Parts (b) and (c) follow from Proposition 3 and the natural weak equivalences $QX \xrightarrow{\sim} X \xrightarrow{\sim} RX$.

Part (3). Let $A \sqcup A \xrightarrow{i_0 \sqcup i_1} A' \xrightarrow{s} A$ be a cylinder object with $s \in W$. Then γs is an isomorphism in $\operatorname{Ho} C$, forcing $\gamma i_0 = \gamma i_1$. For any left homotopy $H\colon A' \to B$ from f to g:

$$\gamma f = (\gamma H)(\gamma i_0) = (\gamma H)(\gamma i_1) = \gamma g.$$

The case of right homotopies follows by duality.

Part (4). Suppose γf is an isomorphism in $\operatorname{Ho} C$. Then QRf is an isomorphism in $C_{cf}/\sim$, hence a homotopy equivalence; by Proposition 6, $QRf \in W$. So $f \in W$. $\square$